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Enhancement of Activated Decay of Metastable States by Resonant Pumping

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We consider the effect of a high-frequency pumping $\mathscr{E} \cos \omega t$ on the escape rate of a classical underdamped Brownian particle out of a deep potential well. The energy dependence of the oscillation frequency $\omega(E)$ is assumed to be weak on the scale of thermal energy, $\omega'_E(0)\, T \sim \omega(0)\, T/V_0 \ll \omega(0)[\omega'_E(0)$ is the derivative of $\omega(E)$ at $E=0$, V_0 is the barrier height, $V_0 \gg T]$. The quadratic-in-$\mathscr{E}$ contribution to the decay rate is calculated in two different regimes: (1) for the case of resonance of the pumping frequency with the nth harmonic of the internal motion at an energy $\tilde{E}$, when $\omega = n\omega(\tilde{E})$; (2) for a rollout region of the basic resonance near the bottom of the potential well, when $|\omega - \omega(0)| \sim \gamma$ and γ is the damping coefficient. In the latter case the absorption spectrum and the enhancement of the decay rate are calculated as functions of two reduced parameters, the anharmonicity of the potential, $v \equiv \omega'_E(0)\, T/\gamma$, and the resonance mismatch, $\delta \equiv [\omega - \omega(0)]/\gamma$. It is shown that the effect of the pumping increases with diminishing $|v|$ and at small v is proportional to v^{-1}. In this regime, the dependence on δ is stepwise: the pumping contribution is large for $v\delta > 0$ and small for $v\delta < 0$. In the frame of our theory, the decay rate is invariant against the simultaneous alternation of the signs of δ and v. The spectrum of the energy absorption has the standard Lorentzian shape in the absence of anharmonicity, $v = 0$, and with increasing of $|v|$ shifts and widens retaining its bell-shape form.

KEY WORDS: Metastable states; activated decay; resonant pumping; energy diffusion; Fokker–Planck equation.

1. INTRODUCTION

Metastable states in deep potential minima have exponentially long lifetimes and decay due to either thermal activation or quantum tunneling. The theoretical investigation of these phenomena has been the subject of

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numerous papers (see, e.g., refs. 1 and 2). Experimentally most promising systems for observation of these processes are Josephson junctions.[3–5] In principle, a Josepson junction works as a nonlinear element of an electrical network. In this function it can be more easily influenced externally than, e.g., chemical systems. The low-lying states of a Josephson junction are described by small oscillations of a particle near the bottom of a potential well. If the damping is weak, one expects that behavior of a Josepson junction under high-frequency pumping will be sensitive to the matching between the frequency of the pumping force and the frequency of an internal motion in the junction. This phenomenon was observed experimentally.[6] Later it was investigated in detail both experimentally and via numerical simulations.[7] Analytical approaches to the solution of this problem were developed for the classical[8] and quantum[9] cases. Still, this problem lacks an exhaustive analytic consideration. For the textbook problem of an oscillator driven by a weak harmonic force at least one final answer is well known: the absorption of energy as a function of the pumping frequency has Lorentzian shape around the oscillator frequency with a width proportional to the damping coefficient. Anharmonicity of the oscillator only becomes important if we consider an ensemble of oscillators at a sufficiently high temperature, when the spread of oscillator frequencies, caused by the thermal spread of the energies, is comparable to the width of the resonance curve.

The aim of this article is to demonstrate that, in accord with the previous findings,[7–9] in the case of a metastable state pumped by a high-frequency force one encounters a qualitatively new physical situation. In contrast to the absorption of energy, to which contribute mainly particles at the bottom of the well, the decay rate is proportional to the magnitude of the distribution function at the barrier top. To reach this energy starting from the particle reservoir at the bottom of the well, a particle must pass through the whole interval of intermediate energies. In the absence of external perturbation, the distribution function is nonequilibrium at the energies close to the top of the barrier due to the escapes of particles across the barrier. In order of magnitude, the typical width of this energy region never exceeds the temperature T. At lower energies the particle distribution retains its Boltzmann shape, so that for a well of a depth V_0 one obtains the well-known Arrhenius law, $1/\tau \propto \exp(-V_0/T)$, where τ is the lifetime of the metastable state. A high-frequency pumping $\mathscr{E} \cos \omega t$ perturbs the particle distribution and in this way affects the decay rate. Below we consider an underdamped particle which, neglecting dissipation and thermal noise, oscillates in the well with a frequency $\omega(E)$ dependent on its energy E. Under these conditions the absorption of energy as a function of the pumping frequency ω has a resonant shape with a width of order

$|\omega-\omega(0)| \sim \gamma$, where γ is the friction coefficient. The energy dependence of the oscillation frequency $\omega(E)$ contributes to widening and shifting of the resonant curve in accord with the spread of oscillation frequencies due to the thermal broadening of the energies, $\omega(E)-\omega(0) \sim \omega'_E(0)\,T$, where $\omega'_E(0)$ is the derivative of $\omega(E)$ at $E=0$. In order of magnitude, $\omega'_E(0) \sim \omega(0)/V_0$, so that $\omega'_E(0)\,T \sim \omega(0)\,T/V_0 \ll \omega(0)$.

We will consider separately the case of resonance of the pumping with an nth harmonic at a certain energy $\tilde{E}$, when

$$n\omega(\tilde{E}) = \omega$$

and the case of basic resonance near the bottom of the well,

$$|\omega-\omega(0)| \sim \gamma$$

In the latter case, our problem will be completely specified by the two dimensionless parameters

$$\delta \equiv \frac{\omega-\omega(0)}{\gamma} \tag{1}$$

$$\nu \equiv \frac{T\omega'_E(0)}{\gamma} \tag{2}$$

where δ determines the reduced mismatch of the resonance, and ν gives the reduced anharmonicity of the potential. In other words, ν determines the interval of energies perturbed by pumping, $\varepsilon \sim T/|\nu|$. We consider the case when effects of the pumping on the dynamics of a particle with an energy near the barrier top, $\varepsilon \sim V_0$, can be neglected. In this case the particle distribution returns to its Boltzmann shape at the energies above the perturbed region. The effect of pumping is manifest then in a change of the magnitude of the Boltzmann function. It is obvious that perturbation of the particle distribution well below the barrier top has no direct influence on the escape processes. This allows us to split the solution of our problem into two stages. In the first stage we calculate the change of the particle distribution caused by pumping. In the second stage we should use the Boltzmann function for the energies above the pumping-perturbed region, $\varepsilon \gg T/|\nu|$, but well below the barrier top, $V_0-\varepsilon \gg T$, as a boundary condition for the solution at the barrier top, when $|V_0-\varepsilon| \sim T$. Fortunately, we do not have to work through this procedure explicitly, since the linearity of the equation for the distribution function allows us to identify the relative change of the decay rate with the relative change of the distribution function in the region of intermediate energies, $\varepsilon \gg T/|\nu|$, $V_0-\varepsilon \gg T$.

The absorption spectrum is determined by particles with thermal energies, as the number of particles drops exponentially with the energy. In contrast, the flux of particles toward the top of the barrier is affected by any perturbations of the distribution function. Therefore, all energies perturbed by the external force contribute practically with equal weight to the change of the decay rate, though the distribution function may change in the actual interval of the energy by several orders of magnitude. As a consequence, it should be expected that the contribution of a high-frequency pumping to the decay rate will be slowly dependent on the pumping frequency as long as it is in resonance with oscillations at a certain energy in the well. At the same time, the pumping-enhanced increase in the decay rate is the greater in magnitude, the smaller the anharmonicity of the potential. For typical potentials, $\omega(E) < \omega(0)$. In this case the resonance conditions will be satisfied for $\omega < \omega(0)$. Correspondingly, in a narrow region, $|\omega - \omega(0)| \sim \gamma$, the pumping contribution to the decay rate drops sharply on going over from a resonant to a nonresonant situation. This qualitative feature of the considered phenomenon was earlier investigated with the use of numerical simulations[7] or analytic calculations.[8,9]

The purpose of this article is the derivation of an exact solution for the quadratic-in-$\mathscr{E}$ contribution to the decay rate in two qualitatively different regimes. First, a resonance with an nth harmonic of the particle motion in the well is considered. The enhancement of the decay rate by the pumping is then obtained as a sum over all harmonics. A more detailed consideration is given for a rolloff region of the resonance, when the resonance mismatch for small oscillations is on the order of the damping coefficient, $|\omega - \omega(0)| \sim \gamma$, i.e., $|\delta| \sim 1$. For a weak anharmonicity, $|\nu| \ll 1$, the solution obtained describes a sharp cutoff of the pumping effects with the transition into a nonresonant regime, $\nu\delta < 0$, and a rather slow dependence on δ in the resonant regime, $\nu\delta > 0$, $|\delta| \gg 1$, where the relative magnitude of the effect is roughly given by $|\nu|^{-1}$. In the regime of a weak friction, $\gamma \ll \omega$, our results can easily be generalized to calculation of the pumping effect in the full region of frequencies, $\omega < \omega(0)$.

2. ENERGY–ANGLE VARIABLES

Our starting point is the Fokker–Planck equation

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial x} + \left(\mathscr{E}\cos\omega t - \frac{\partial V(x)}{\partial x}\right)\frac{\partial F}{\partial p} = \gamma\frac{\partial}{\partial p}\left(mT\frac{\partial F}{\partial p} + pF\right) \tag{3}$$

where $F(p, x, t)$ is the distribution function, x, p, and m are, respectively, the coordinate, momentum, and mass of a particle, $V(x)$ is a potential of finite depth with the minimal value $V = 0$ at $x = 0$, $\mathscr{E}\cos\omega t$ is an external force, and γ is the damping coefficient. The temperature T is considered to be small compared to the scale of the potential energy, $T \ll V(x)$, so that we can use a harmonic approximation near the bottom of the well. In this case the normalized equilibrium function is given by

$$F_0(p, x) = \frac{1}{2\pi\omega(0)\, T} \exp\left(-\frac{E}{T}\right) \tag{4}$$

where E is the total energy,

$$E = \frac{p^2}{2m} + V(x) \tag{5}$$

and $\omega(0)$ is the frequency of small oscillations,

$$\omega(0) \equiv [V''(0)/m]^{1/2}$$

The motion in the well is assumed to be underdamped, in other words, γ is small compared to ω, which is close to the frequency $\omega(0)$. Due to anharmonicity of the potential, only energies near the bottom of the well are perturbed by pumping. This assumption simplifies our task substantially, since under these conditions the distribution function perturbed by the pumping in a certain interval of energies recovers its Boltzmann shape at higher energies. The relative change of its magnitude gives then directly the relative change of the decay rate. In the absence of an external force, friction, and thermal noise, the motion of a particle in a potential well is completely specified by the conserved energy E and the angle $\varphi(t) = \omega(E)\, t + \varphi_0$. As long as they are weak, an external force, damping, and thermal noise cause slow changes in the energy E and the phase φ_0. In order to write down the Fokker–Planck equation in terms of these slowly varying quantities, we consider first the dynamic equations

$$\frac{dp}{dt} = -\frac{\partial V(x)}{\partial x} + \mathscr{E} \cos \omega t \tag{6}$$

$$\frac{dx}{dt} = \frac{p}{m} \tag{7}$$

and transform them to the variables E, φ. The influence of the damping and thermal noise will then be accounted for in a standard way. To map the variables (p, x) onto the variables (E, φ), we make use of the solutions of Eqs. (6) and (7) in the absence of an external force, $\mathscr{E} = 0$,

$$p(t) = p(E, \varphi) \tag{8}$$

$$x(t) = x(E, \varphi) \tag{9}$$

In other words, $x(E, \varphi)$ with $\varphi = \omega(E)\,t + \text{const}$ describes unperturbed motion with the energy E. The explicit expressions for $p(E, \varphi)$, $x(E, \varphi)$, and $\omega(E)$ in the first-order approximation at $E \ll V_0$ are given in ref. 10. In terms of the functions (8) and (9) the equations for E and φ are[11,12]

$$\frac{dE}{dt} = \frac{p(E, \varphi)}{m}\, \mathscr{E} \cos \omega t \tag{10}$$

$$\frac{d\varphi}{dt} = \omega(E) - \omega(E)\, \frac{\partial x(E, \varphi)}{\partial E}\, \mathscr{E} \cos \omega t \tag{11}$$

These equations have in their right-hand sides either slowly varying or small and rapidly oscillating terms. Keeping in mind that the typical scale of time in our problem is of the order of $\gamma^{-1} \gg \omega^{-1}$, one can average Eqs. (10) and (11) over an interval of time $\gamma^{-1} \gg \Delta t \gg \omega^{-1}$, getting as a result equations without an explicit dependence on time.

3. THE FOKKER–PLANCK EQUATION FOR SLOW VARIABLES

To eliminate the fast dependence on time in the Fokker–Planck equation (3), we introduce the following Fourier expansions:

$$x(E, \varphi) = \sum_{n=1}^{\infty} x_n(E) \sin(n\varphi)$$

$$p(E, \varphi) = \sum_{n=1}^{\infty} p_n(E) \cos(n\varphi)$$

where the coefficients $x_n(E)$ and $p_n(E)$ are connected by the relation

$$p_n(E) = n\omega(E)\, x_n(E)$$

If the external pumping $\mathscr{E} \cos \omega t$ is in resonance with the nth harmonic at an energy E,

$$|\omega - n\omega(E)| \ll \omega$$

one can retain in Eqs. (10) and (11) only resonant terms, so that after the substitution

$$\varphi(t) \to \varphi(t) + \omega t/n$$

and averaging of the equations over a period of pumping oscillation, Eqs. (10) and (11) yield

$$\begin{aligned}\frac{dE}{dt}&=\frac{p_n(E)}{2m}\,\mathscr{E}\cos n\varphi\\ \frac{d\varphi}{dt}&=\omega(E)-\frac{\omega}{n}-\frac{1}{2}\,\omega(E)\,\frac{\partial x_n(E)}{\partial E}\,\mathscr{E}\sin n\varphi\end{aligned}\tag{12}$$

In these equations the small parameters of our problem, the force amplitude $\mathscr{E}$ and the frequency mismatch $\omega(E)-\omega$, enter on an equal basis. Generalization of these equations to account for the damping and thermal noise is straightforward as both these factors are additive with the force $\mathscr{E}\cos\omega t$ in Eqs. (10) and (11). To proceed further, we write down the Fokker–Planck equation in the variables E, φ,

$$\begin{aligned}&\frac{\omega(E)-\omega/n}{\gamma}\frac{\partial F}{\partial\varphi}+\frac{\mathscr{E}}{2\gamma}\left[\frac{p_n(E)}{m}\cos\varphi\,\frac{\partial F}{\partial E}-\omega(E)\,\frac{\partial x_n(E)}{\partial E}\,\frac{\partial F}{\partial\varphi}\right]\\ &=\frac{\partial}{\partial E}\,m\overline{v^2}(E)\left(T\frac{\partial F}{\partial E}+F\right)+D_\varphi(E)\,\frac{\partial^2 F}{\partial\varphi^2}\end{aligned}\tag{13}$$

The right-hand side of this equation accounts for the effects of noise and dissipation. It is written taking into account the relations

$$\begin{aligned}\langle\Delta E\rangle&=-\gamma m\overline{v^2}(E)\,\Delta t\\ \langle\Delta\varphi\rangle&=0\\ \langle(\Delta E)^2\rangle-\langle\Delta E\rangle^2&=\gamma m\overline{v^2}(E)\,T\Delta t\\ \langle(\Delta\varphi)^2\rangle&=2D_\varphi(E)\,\Delta t\\ \langle\Delta E\Delta\varphi\rangle&=0\end{aligned}\tag{14}$$

where ΔE and $\Delta\varphi$ are variations of E and φ over the interval of time Δt, $\gamma^{-1}\gg\Delta t\gg\omega^{-1}$, caused by the damping and thermal noise,

$$\overline{v^2}(E)\equiv\int_0^{2\pi}\frac{d\varphi}{2\pi}\,v^2(E,\varphi)$$

and $D_\varphi(E)$ is the angle diffusion coefficient,

$$D_\varphi(E)\equiv m\gamma T\omega^2(E)\int_0^{2\pi}\frac{d\varphi}{2\pi}\left(\frac{\partial x(E,\varphi)}{\partial E}\right)^2$$

To derive the above relations, one needs to consider the averaging of Eqs. (10) and (11) with $\mathscr{E}\cos\omega t$ substituted by $\mathscr{E}\cos\omega t-\gamma p[E,\omega(E)\,t]+\eta(t)$, where $\eta(t)$ is the Gaussian thermal noise with the correlator

$$\langle\eta(t)\,\eta(t')\rangle=2m\gamma\delta(t-t')$$

Below, we shall distinguish between resonance at intermediate energies, when $n\omega(\tilde{E}) = \omega$ at a certain energy $\tilde{E} \gg T$ (see the next section), and the resonance at the bottom of the well, when $|\omega - \omega(0)| \ll \omega$ (see Section 5).

4. RESONANCES AT INTERMEDIATE ENERGIES

For $E \gg T$, one should neglect the terms with second derivatives in Eq. (13) which are small in the parameter $T/E \ll 1$. Substitution of the expansion

$$F(E, \varphi) = F_0(E)\left\{1 + \mathscr{E}\,\mathrm{Re}[f_1(E)\,e^{i\varphi}] + \frac{\mathscr{E}^2}{2} f_2(E)\right\} \tag{15}$$

into the resulting equation, yields the system

$$-i\frac{n\omega(E)-\omega}{\gamma} f_1(E) - m\overline{v^2}(E)\frac{df_1(E)}{dE} = -\frac{1}{\gamma}\frac{n\omega(E)\,x_n(E)}{2T}$$

$$m\overline{v^2}(E)\,T\frac{df_2(E)}{dE} = \frac{n\omega(E)}{2\gamma}\frac{\partial x_n(E)}{\partial E}\,\mathrm{Re}\,f_1(E)$$

In all the coefficients of these equations, with the only exception of the difference $n\omega(E) - \omega$, one can substitute E by the constant $\tilde{E}$. With the use of the linear expansion

$$n\omega(E) - \omega \approx n\omega'(\tilde{E})(E - \tilde{E})$$

for the function $f_1(E)$, one obtains

$$f_1(E) = \frac{\omega(\tilde{E})\,x_n(\tilde{E})}{2\omega'(\tilde{E})\,Tw(\tilde{E})}\exp\left[-\frac{i(E-\tilde{E})^2}{2w(\tilde{E})}\right]\int_{-\infty}^{E}\exp\left[\frac{i(E'-\tilde{E})^2}{2w(\tilde{E})}\right]dE' \tag{16}$$

where

$$w(\tilde{E}) \equiv \frac{\gamma m\overline{v^2}(\tilde{E})}{n\omega'(\tilde{E})}$$

From this expression it follows that $f_1(E)$ is concentrated around $E = \tilde{E}$ with a half-width of the order $w^{1/2}(\tilde{E})$.

It follows then that the neglect of the terms with second derivatives in Eq. (13) is only justified for $w^{1/2} \gg T$, in other words, for

$$\gamma/\omega \gg (T/V_0)^2$$

This criterion is not extremely severe, since it is compatible with the criterion $\gamma/\omega \ll 1$, which provides a condition of high-quality oscillations, and with the criterion $\gamma/\omega \sim T/V_0$, under which the results of Section 6 are applicable. The relative change of the distribution function at the energies above the perturbed region is given by the quantity $(1/2)\,\mathscr{E}^2 f_2(\infty)$. The distribution function

$$F(\varepsilon) \equiv F_0(\varepsilon)\left[1 + \frac{\mathscr{E}^2}{2} f_2(\infty)\right]$$

for the energies above the perturbed region but below the barrier top, i.e., for $V_0 - \varepsilon \gg T$, must be used as a boundary condition deep in the well when solving the problem of the escapes over the barrier at $|V_0 - \varepsilon| \sim T$. However, it is obvious that the relative change in the decay rate is the same as the relative change in the distribution function. For the decay rate one then finds

$$\frac{D(\mathscr{E})}{D(0)} = 1 + \frac{\mathscr{E}^2}{2} f_2(\infty) \equiv 1 + \frac{\mathscr{E}^2}{2m\gamma^2 T} \mathscr{K}_n(\omega)$$

Substitution of expression (16) into the equation for $f_2(E)$ and its solution with the boundary condition

$$f_2(E) = 0, \qquad \tilde{E} - E \gg w^{1/2}(\tilde{E})$$

yields finally

$$\mathscr{K}_n(\omega) = \frac{\pi}{2} \frac{n\gamma m\omega^2}{T\omega'(E)} \frac{\partial}{\partial E} x_n^2(E)\bigg|_{\omega(E) = \omega/n} \tag{17}$$

It is worth noting that the quantity $\overline{v^2}(\tilde{E})$ cancels out in the relation for $\mathscr{K}_n$. The result obtained coincides with that derived by Larkin and Ovchinnikov[9] for $n = 1$. Recently, the pumping effects on nonlinear oscillations and enhancement of the decay rate were considered by Linqwitz and Grabert.[13] The total contribution of a pumping into the decay rate is given by the sum over all resonances,

$$\mathscr{K}(\omega) = \sum_{n=0}^{\infty} \mathscr{K}_n(\omega)$$

For the basic resonance ($n = 1$) near the bottom of the potential well, when $E \ll V_0$, one must substitute into Eq. (17) the approximate relation

$$x_1(E) \approx \frac{1}{\omega(0)} \left(\frac{2E}{m}\right)^{1/2} \tag{18}$$

which gives

$$\mathscr{K}_1(\omega) = \frac{\pi}{|\nu|}\,\theta[\omega(0) - \omega] \tag{19}$$

It is assumed here that the frequency $\omega(E)$ is decreasing with E, so that $\omega(E) < \omega(0)$. In the next section the rolloff region is considered, where $\mathscr{K}_1(\omega)$ is smoothly changing from the value $\pi/|\nu|$ to a vanishing value in the interval of frequencies $|\omega - \omega(0)| \sim \gamma$.

5. THE ROLLOFF REGION OF THE BASIC RESONANCE

To investigate the range of frequencies $|\omega(0) - \omega| \sim \gamma$, one has to retain all terms in Eq. (13). It is convenient to introduce the function

$$F(E, \varphi) = F_0(\varepsilon)\left\{1 + \tilde{\mathscr{E}}\,\mathrm{Re}[f_1(\varepsilon)\,e^{i\varphi}] + \frac{\tilde{\mathscr{E}}^2}{2} f_2(\varepsilon)\right\} \tag{20}$$

where $F_0(\varepsilon)$ is given by Eq. (4), and we have introduced the dimensionless energy $\varepsilon \equiv E/T$ and dimensionless amplitude of the pumping force,

$$\tilde{\mathscr{E}} \equiv \frac{\mathscr{E}}{\gamma(2mT)^{1/2}}$$

Averaging of the right-hand side of Eq. (10) with the distribution function (20) gives the rate of the energy loss,

$$P \equiv \langle dE/dt \rangle = \gamma T(\tilde{\mathscr{E}}^2/2)\,S(\delta, \nu)$$

where $S(\delta, \nu)$ is the spectrum of absorption,

$$S(\delta, \nu) = \mathrm{Re}\int_0^\infty e^{-\varepsilon} f_1(\varepsilon)\,\varepsilon^{1/2}\,d\varepsilon$$

The complex function $f_1(\varepsilon)$ is governed by the equation

$$\varepsilon\frac{d^2 f_1}{d\varepsilon^2} + (1 - \varepsilon)\frac{df_1}{d\varepsilon} + \left(i\delta - i\nu\varepsilon - \frac{1}{4\varepsilon}\right) f_1 = -\varepsilon^{1/2}$$

which follows from Eq. (13) for $n = 1$ with account being taken of Eq. (18) and notations (1) and (2). The second-order correction to the distribution function obeys the equation

$$\varepsilon\frac{df_2(\varepsilon)}{d\varepsilon} = \varepsilon^{1/2}\,\mathrm{Re}\,f_1(\varepsilon) \tag{21}$$

This equation follows from the second-order equation for $f_2(\varepsilon)$, if we drop out the term $C\exp(\varepsilon)$ in the right-hand side, which for finite C gives a logarithmic divergency of $f_2(\varepsilon)$ at $\varepsilon \to 0$. The general solution of Eq. (21) is

$$f_2(\varepsilon) = f_2(\infty) - \mathrm{Re} \int_{\varepsilon}^{\infty} x^{-1/2} f_1(x)\, dx$$

Perturbation of the particle distribution by a pumping does not change the total number of the particles. Therefore, the integration constant $f_2(\infty)$ must be found from the notmalization condition for the function (20), which is equivalent to

$$\int_0^{\infty} e^{-\varepsilon} f_2(\varepsilon)\, d\varepsilon = 0$$

The relative change of the distribution function at the energies above the perturbed region is given by the quantity $(1/2)\,\mathscr{E}^2 f_2(\infty)$. This means that we need to calculate the quantity[8]

$$f_2(\infty) = \mathrm{Re} \int_0^{\infty} \frac{[1 - \exp(-\varepsilon)]\, f_1(\varepsilon)\, d\varepsilon}{\varepsilon^{1/2}}$$

For the decay rate one then finds

$$\frac{D(\tilde{\mathscr{E}})}{D(0)} = 1 + \frac{\tilde{\mathscr{E}}^2}{2} f_2(\infty) \equiv 1 + \frac{\tilde{\mathscr{E}}^2}{2} \mathscr{K}(\delta, v)$$

Substitution of

$$f_1(\varepsilon) = \varepsilon^{-1/2} z(\varepsilon)$$

reduces the expression for $\mathscr{K}(\delta, v)$ to

$$\mathscr{K}(\delta, v) = \mathrm{Re} \int_0^{\infty} \frac{[1 - \exp(-\varepsilon)]\, z(\varepsilon)\, d\varepsilon}{\varepsilon}$$

with the function $z(\varepsilon)$ obeying the equation

$$\varepsilon z'' - \varepsilon z' + (i\delta + 1/2 - iv\varepsilon)\, z = -\varepsilon \tag{22}$$

It should be noted that the parameters δ and v enter this equation with the factor i, whereas the physical quantities are determined by the real part of $z(\varepsilon)$. This means that the functions $S(\delta, v)$ and $\mathscr{K}(\delta, v)$ depend on the combinations δ^2, ν^2, and $\delta\nu$, in other words, these functions do not change by simultaneous alternation of the signs of δ and ν.

If $|\nu| \ll 1$, the main contribution to $\mathscr{K}(\delta, \nu)$ comes from $\varepsilon \sim 1/|\nu| \gg 1$. One must therefore drop the term with second derivative in Eq. (22), obtaining the equation

$$-\varepsilon z' + (i\delta - i\nu\varepsilon + 1/2)\, z = -\varepsilon \tag{23}$$

which can be solved straightforwardly,

$$z = -\varepsilon^{i\delta + 1/2} e^{-i\nu\varepsilon} \int_{\varepsilon}^{\infty} e^{i\nu\varepsilon'} \varepsilon'^{-i\delta - 1/2}\, d\varepsilon' \tag{24}$$

For $\mathscr{K}(\nu, \delta)$ it then follows that

$$\mathscr{K}(\nu, \delta) = \frac{1}{\nu} \lim_{\varepsilon \to \infty} \operatorname{Im} \int_1^{\infty} \frac{x^{-i\delta - 1/2}}{x - 1} [1 - e^{i\nu\varepsilon(x-1)}]\, dx$$

where we have interchanged the order of integration. Finally, this equation yields

$$\mathscr{K}(\nu, \delta) = \frac{1}{\nu}\left[\operatorname{Im} \psi\left(i\delta + \frac{1}{2}\right) + \frac{\pi}{2} \operatorname{sign} \nu\right] = \frac{\pi}{|\nu|} \frac{1}{1 + \exp(-2\pi\delta \operatorname{sign} \nu)} \tag{25}$$

This expression describes in more detail a narrow region of frequencies approximated earlier by the step function (19).

To solve Eq. (22) in the general case, one has to use a more powerful approach. From the condition of boundedness of $f_2(\varepsilon)$ at $\varepsilon = 0$ we conclude that $z(0) = 0$. At small ε the function $z(\varepsilon)$ is expandable in powers of ε. In this way we obtain the asymptotic behavior of $z(\varepsilon)$,

$$z \approx \frac{\varepsilon(1 + 2a)}{1/2 - i\delta} + a\varepsilon^2 + \ldots, \qquad \varepsilon \ll 1$$

The unknown parameter a has to be chosen in such a way as to kill the amplitude A by the growing exponent in the general solution of Eq. (22),

$$z \approx A \exp(\varepsilon/2 + \rho\varepsilon) + B \exp(\varepsilon/2 - \rho\varepsilon) + 1/i\nu; \qquad \varepsilon \gg 1 \tag{26}$$

where the exponential terms describe a solution of the homogeneous equation,

$$\rho \equiv (\tfrac{1}{4} + i\nu)^{1/2}$$

Equation (22) can be solved exactly with the use of the Laplace transformation,

$$\varphi(\lambda) \equiv \int_0^\infty e^{-\lambda\varepsilon - \varepsilon/2} z(\varepsilon)\, d\varepsilon$$

The functions $S(\delta, v)$ and $\mathscr{K}(\delta, v)$ in terms of the function $\varphi(\lambda)$ are then given by

$$S(\delta, v) = \operatorname{Re} \int_0^\infty e^{-\varepsilon} z(\varepsilon)\, d\varepsilon = \operatorname{Re}\, \varphi(1/2)$$

$$\mathscr{K}(\delta, v) = \int_{-1/2}^{1/2} \varphi(\lambda)\, d\lambda$$

The new function $\varphi(\lambda)$ obeys the first-order differential equation

$$\frac{d}{d\lambda}(\lambda^2 - \rho^2)\,\varphi - \left(i\delta + \frac{1}{2}\right)\varphi = \frac{1}{(\lambda + 1/2)^2} \tag{27}$$

The solution of the homogeneous equation is

$$\varphi_0(\lambda) = \frac{1}{\lambda^2 - \rho^2}\left(\frac{\lambda - \rho}{\lambda + \rho}\right)^\mu$$

where

$$\mu \equiv \frac{i\delta + 1/2}{2\rho} = \frac{i\delta + 1/2}{(1 + 4iv)^{1/2}}$$

This function has two singular points, $\lambda = \pm\rho$, which correspond to two exponents in the asymptotics (26). As explained above, we demand the coefficient A in Eq. (26) be vanishing. The solution of the inhomogeneous equation (27) is then determined by the condition of analyticity of $\varphi(\lambda)$ at the point $\lambda = \rho$,

$$\varphi(\lambda) = -\frac{1}{\lambda^2 - \rho^2}\left(\frac{\lambda - \rho}{\lambda + \rho}\right)^\mu \int_\lambda^\rho \left(\frac{\lambda' + \rho}{\lambda' - \rho}\right)^\mu \frac{d\lambda'}{(\lambda' + 1/2)^2}$$

Now we get the explicit expression for the absorption spectrum (see also ref. 14),

$$S(\delta, v) = \frac{1}{v} \operatorname{Im} \left(\frac{1/2 - \rho}{1/2 + \rho}\right)^\mu \int_{1/2}^\rho \left(\frac{\lambda' + \rho}{\lambda' - \rho}\right)^\mu \frac{d\lambda'}{(\lambda' + 1/2)^2}$$

which can be reduced to a simpler form,

$$S(\delta, \nu) = \frac{1}{\nu} \operatorname{Im} \frac{1+s}{\rho + 1/2} \sum_{n=1}^{\infty} \frac{n s^{2n-1}}{n - \mu}$$

where

$$s \equiv \frac{\rho - 1/2}{\rho + 1/2} = \frac{(1+4i\nu)^{1/2} - 1}{(1+4i\nu)^{1/2} + 1}$$

For small anharmonicity, $|\nu| \ll 1$, the standard Lorentz expression is reproduced,

$$S(\delta, \nu) \approx \frac{2}{1 + 4\delta^2}$$

The detailed exposition of the derivation for $\mathscr{K}(\delta, \nu)$ is given elsewhere.[12] It yields the following result:

$$\begin{aligned}\mathscr{K}(\delta, \nu) = &-\frac{1}{\nu} \operatorname{Im} \left\{ \psi \left[1 - \frac{i\delta + 1/2}{(1+4i\nu)^{1/2}} \right] \right. \\ &\left. + \sum_{n=1}^{\infty} \left[\frac{(1+4i\nu)^{1/2} - 1}{(1+4i\nu)^{1/2} + 1} \right]^{2n} \frac{1}{n - \mu} \right\} \\ &+ \frac{\pi}{2|\nu|} - \frac{1}{2\nu} \arcsin \frac{4\nu}{(1+16\nu^2)^{1/2}}\end{aligned} \tag{28}$$

where $\psi(x)$ is the Euler psi function. In the most interesting case of a weak anharmonicity of the potential, $|\nu| \ll 1$, the first two terms of the expansion in this small parameter are

$$\mathscr{K}(\delta, \nu) \approx \frac{\pi}{|\nu| \, [1 + \exp(-2\pi\delta \operatorname{sign} \nu)]} + 2 \operatorname{Re} \left(i\delta - \frac{1}{2} \right) \psi' \left(i\delta + \frac{1}{2} \right) - 2$$

where the second term serves as a correction to Eq. (25). In the nonresonant region, $-\delta \operatorname{sign} \nu \gg 1$, one gets

$$\mathscr{K}(\delta, \nu) \approx \frac{1}{6\delta^2}$$

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